$\forall k \in \mathbb{N}, \quad \sqrt{k+2} \geq \sqrt{k+1}$

$\Rightarrow \sqrt{k+1} + \sqrt{k+2} \geq 2\sqrt{k+1}$

$\Rightarrow \dfrac{1}{2\sqrt{k+1}} \geqslant \dfrac{1}{\sqrt{k+1}+\sqrt{k+2}}$

D'où $\boxed{\forall k \in \mathbb{N}; \sqrt{k+2} - \sqrt{k+1} \leqslant \dfrac{1}{2\sqrt{k+1}}}$

c) $\forall n \in \mathbb{N}; \quad S_n = \sum_{k=0}^{n} u_k = \sum_{k=0}^{n} \dfrac{1}{\sqrt{k+1}}$

Donc $\forall k \in \mathbb{N}; \sqrt{k+2} - \sqrt{k+1} \leqslant \dfrac{1}{2} u_k$

$\Rightarrow \sum_{k=0}^{n} \left(\sqrt{k+2} - \sqrt{k+1}\right) \leqslant \dfrac{1}{2} \sum_{k=0}^{n} u_k$

$\Rightarrow (\sqrt{2} - 1) + (\sqrt{3} - \sqrt{2}) + \dots + (\sqrt{n+2} - \sqrt{n+1}) \leqslant \dfrac{1}{2} S_n$

$\Rightarrow \sqrt{n+2} - 1 \leqslant \dfrac{1}{2} S_n$

$\Rightarrow \boxed{S_n \geqslant 2(\sqrt{n+2} - 1) \quad \forall n \in \mathbb{N}}$

•) $\lim_{n \to +\infty} 2(\underbrace{\sqrt{n+2}}_{\to +\infty} - 1) = +\infty$

Ainsi $\boxed{\lim_{n \to +\infty} S_n = +\infty}$

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